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LETTER TO THE EDITOR

Spectrum of a spin chain with inverse square exchange

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Abstract. The spectrum of a one-dimensional chain of $SU(n)$ spins positioned at the static equilibrium positions of the particles in the corresponding classical Calogero system with an exchange interaction inversely proportional to the square of their distance is studied. As in the translationally invariant Haldane–Shastry model the spectrum is found to exhibit a very simple structure containing highly degenerate ‘super-multiplets’. The algebra underlying this structure is identified and several sets of raising and lowering operators are given explicitly. On the basis of this algebra and numerical studies we give the complete spectrum and thermodynamics of the $SU(2)$ system.

Since it was first introduced [1, 2] the Haldane–Shastry model for spin chains with inverse square exchange and its generalizations to $SU(n)$ spins have attracted considerable interest [3–11]. The Hamiltonian of these systems is given by the expression

$$\mathcal{H} = \sum_{j < k} h_{jk} P_{jk} \quad (1)$$

where $h_{jk} = \sin^{-2}((\pi/L)(x_j - x_k))$ is the exchange constant for spins at lattice sites with coordinates x_j and x_k and P_{jk} is the operator that exchanges the spins at these sites. The x_j are chosen to be equally spaced around a ring of circumference L . Haldane and Shastry have given the wavefunctions for the antiferromagnetic ground state which is of product form—just as in the Calogero–Sutherland model of particles on a line interacting with inverse square potential—and also found the possible energy levels for the system. The spectrum of this model exhibits a very simple structure including a highly degenerate ‘super-multiplet’ structure. Very recently, this structure has been shown to be the consequence of a Yangian symmetry [12, 13] present in this model [9]. This symmetry which is intimately related to the integrability of a system may eventually provide a way to construct a generating function for the conserved quantities [14], equivalent to the transfer matrix in the nearest neighbour exchange models that are soluble by the quantum inverse scattering method [15], however this has not been achieved yet.

A complete set of these conserved quantities for the Haldane–Shastry $SU(n)$ chain has been found by Fowler and Minahan [10]. Using the exchange operator formalism [16] developed recently to study the integrability of the Calogero–Sutherland models (see e.g. [17–22]), they succeeded in constructing operators commuting with \mathcal{H} provided that the lattice sites x_j are equally spaced.

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This work has been generalized by Polychronakos [23] to construct a new integrable model of a spin chain with exchange interaction related to another one of the classical Calogero–Sutherland models. In a calculation completely analogous to that of Fowler and Minahan, he was able to construct a complete set of invariants for the Hamiltonian (1) with exchange coupling

$$h_{jk} = \frac{1}{(x_j - x_k)^2} \quad (2)$$

provided the x_j are the static equilibrium positions of particles in the classical N -body Calogero system with potential [17, 18]

$$V(x_1, \dots, x_N) = \frac{1}{2} \sum_j x_j^2 + \sum_{j < k} \frac{1}{(x_j - x_k)^2} \quad (3)$$

namely

$$x_j = \sum_{k \neq j} \frac{2}{(x_j - x_k)^3}. \quad (4)$$

(The same requirement leads to the lattice of equally spaced x_j in the Haldane–Shastry model.)

While this construction of complete sets of commuting integrals provides a proof for the integrability of the models (1) it does not give any hint on how to construct eigenstates and thus study the energy spectrum. This is in contrast to the model with nearest neighbour exchange where the algebraic Bethe ansatz provides the complete solution of this problem [15]. For some of the Calogero–Sutherland models [20, 22] and the $SU(2)$ Haldane–Shastry model [1] it has been shown that an *asymptotic* Bethe ansatz reproduces the exact spectrum and thermodynamics. However, a rigorous derivation of the corresponding equations is still lacking.

The purpose of this letter is to present results of a study of the spectrum of the integrable spin chain (1) with inverse square exchange (2). Raising and lowering operators connecting states of different energy are constructed which explain the simple spectrum of this model and allows one to obtain explicit expressions for some eigenstates for the $SU(2)$ chain—including the antiferromagnetic ground state which is again of product form. Finally, an effective Hamiltonian is given that reproduces the complete spectrum and degeneracies of the spin chain Hamiltonian. In a similar way to [24] this allows one to study the thermodynamics of the system.

At first sight, the condition (4) fixing the positions x_j of spins appears to be a major obstacle to the study of the spectrum of this system with analytical methods. Fortunately, the solutions x_j of this equation can be identified as being the N roots of the Hermite polynomial $H_N(x)$ of degree N . In fact, it is straightforward to obtain (4) from the differential equation defining $H_N(x)$, namely $y'' - 2xy' + 2Ny = 0$ (see e.g. [25]). More relations between the x_j of type (4) that are useful in computing certain properties of the system can be found in an analogous way, e.g.

$$x_j = \sum_{k \neq j} \frac{1}{x_j - x_k} \quad 2(N - 1) - x_j^2 = \sum_{k \neq j} \frac{1}{(x_j - x_k)^2}. \quad (5)$$

For large N the density of roots of H_N and hence the density of sites in this non-uniform lattice is known to be

$$\rho_N(x) = \frac{1}{\pi} \sqrt{2N + 1 - x^2}. \tag{6}$$

To proceed, we choose a representation of the permutation operator in (1) in terms of $SU(n)$ spin-operators J_i^α defined to be the $n^2 - 1$ traceless Hermitian $n \times n$ -matrices of the fundamental representation of the Lie algebra $[J^\alpha, J^\beta] = f^{\alpha\beta\gamma} J^\gamma$ normalized such that $\text{Tr}(J^\alpha J^\beta) = \frac{1}{2} \delta^{\alpha\beta}$. The $f^{\alpha\beta\gamma}$ are the antisymmetric structure constants of $SU(n)$ and J_i is understood as acting in the space of the spin at site i only. For $n = 2$, $f^{\alpha\beta\gamma} = i\epsilon^{\alpha\beta\gamma}$ and $2J_i^\alpha$ are Pauli-matrices. Through these spin-operators the permutation operator P_{jk} can be expressed as

$$P_{jk} = \frac{1}{n} \mathbf{1}_j \otimes \mathbf{1}_k + 2 \sum_{\alpha} J_j^\alpha \otimes J_k^\alpha \tag{7}$$

($\mathbf{1}_j$ is the $n \times n$ unit matrix acting in the Hilbert space of the spin at site j). From this one has to expect the eigenvalues of (1) to be grouped into $SU(n)$ multiplets since

$$[\mathcal{H}, Q_0^\alpha] = 0 \quad \alpha = 1, \dots, n^2 - 1 \tag{8}$$

with $Q_0^\alpha = \sum_j J_j^\alpha$ being the components of the total spin.

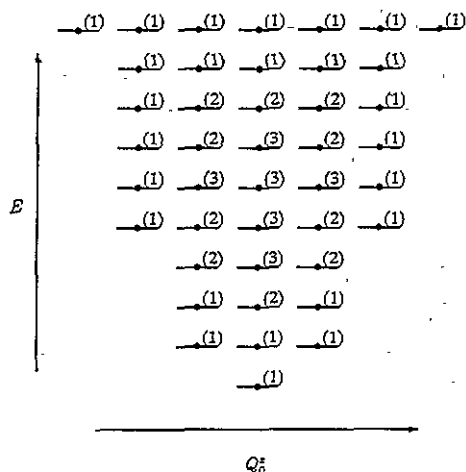


Figure 1. Spectrum of the spin chain (1) with exchange (2) for $N = 6$ spins (energy against z -component of total spin Q_0^z). The degeneracies are given in parentheses.

However, numerical diagonalization of the Hamiltonian \mathcal{H} for $SU(2)$ spins with exchange coupling (2) on small lattices ($N \leq 10$) satisfying (4) shows—very similar to the observations in the Haldane–Shastry model—a much simpler structure (see figure 1). In addition to the expected $SU(2)$ multiplets one finds that all the energies are integers, and that there are additional degeneracies between states having different total spin.

Similar ‘super-multiplets’ have been observed in the periodic $SU(2)$ Haldane–Shastry model [1,2]. The generators of the algebra corresponding to this symmetry have been

identified in [9]. Following this work, we are led to study the commutators of the Hamiltonian (1) with the operators

$$L_1^\alpha = \sum_{j \neq k} w_{jk} f^{\alpha\beta\gamma} J_j^\beta J_k^\gamma. \quad (9)$$

It turns out that choosing $w_{jk} = 1/(x_j - x_k)$ leads to the vanishing of terms containing spin-operators at three different sites. Furthermore, since the J_j^α act in the fundamental representation, one finds using (4) that†

$$[\mathcal{H}, L_1^\alpha] = L_0^\alpha \equiv \sum_{j=1}^N x_j J_j^\alpha. \quad (10)$$

Similarly, one obtains $[\mathcal{H}, L_0^\alpha] = L_1^\alpha$ from which $n^2 - 1$ pairs of raising (lowering) operators can be defined with

$$Q_\pm^\alpha = L_0^\alpha \pm L_1^\alpha. \quad (11)$$

It is easily checked that

$$[Q_0^\alpha, L_i^\beta] = f^{\alpha\beta\gamma} L_i^\gamma \quad i = 0, 1. \quad (12)$$

By construction an equivalent relation holds for the commutator $[Q_0^\alpha, Q_\pm^\beta]$.

This observation allows one to study the spectrum of the Hamiltonian (1) in more detail. To be specific we consider the case of $SU(2)$ corresponding to an isotropic spin- $\frac{1}{2}$ chain in the following. From $Q_0^z |0; m\rangle = 0$ one easily finds for the state of highest energy with $Q_0^z |0; m\rangle = (\frac{1}{2}N - m) |0; m\rangle$ ($|0\rangle = |\uparrow_1 \cdots \uparrow_N\rangle$) is the ferromagnetic vacuum

$$|0; m\rangle \propto \sum_{j_1 < j_2 < \cdots < j_m} \psi_{j_1 j_2 \cdots j_m} \prod_{s=1}^m J_{j_s}^- |0\rangle \quad (13)$$

($J_j^\pm = J_j^x \pm iJ_j^y$) with $\psi_{j_1 \cdots j_m} \equiv 1$. This is the multiplet with total spin $S = N/2$, i.e. all ferromagnetic states $|0; m\rangle = (Q_0^-)^m |0\rangle$ in the system. Acting on $|0; m\rangle$ with Q_-^z and $Q_-^\pm = Q_-^x \pm iQ_-^y$ eigenstates with lower energies can be constructed. For example, the states with a single overturned spin as compared to the ferromagnetic vacuum $|0\rangle$, namely $(Q_-^z)^n |0; 1\rangle$, $n = 0, \dots, N - 1$ are of the form (13) with

$$\psi_j^{(n)} = \pi^{(n)}(x_j). \quad (14)$$

Here $\pi^{(n)}(x)$ is a polynomial in x of degree n , belonging to the set of orthogonal polynomials on the discrete set $\{x_j\}$ with unit weight function defined through the recursion relations

$$\pi^{(n)}(x) = A_n x \pi^{(n-1)}(x) - C_n \pi^{(n-2)}(x) \quad n = 1, \dots, N - 1 \quad (15)$$

with $\pi^{(-1)} \equiv 0$, $\pi^{(0)}(x) \equiv 1/\sqrt{N}$ and

$$A_n = \sqrt{\frac{2}{N-n}} \quad C_n = \frac{A_n}{A_{n-1}}. \quad (16)$$

† For the $SU(n)$ Haldane-Shastry model an operator of the form (9) has been shown to commute with \mathcal{H} [9].

Similarly, it can be shown, that each state with energy $-n$ as compared with the ferromagnetic state and magnetization $\frac{1}{2}N - m$ can be written as a polynomial in m variables $y_j \in \{x_k\}$, $j = 1, \dots, m$ where the powers in the terms $\prod_j (y_j)^{m_j}$ satisfy $\sum_j m_j \leq n$.

Of course, this form of the eigenstates is not very useful when one is interested in studying the properties of the antiferromagnetic ground state with $m = \frac{1}{2}N$ and energy $E_0 = -(\frac{1}{2}N)^2$ (for an even number of lattice sites N). For the Haldane-Shastry model it is known that the ground state can be written in product form. The same turns out to be true for this model. One finds that the lowest state with total magnetization $N/2 - m$ can be written in the form (13) with

$$\psi_{j_1 \dots j_m} = \prod_{j \in \{j_\alpha\}} (-1)^j \prod_{j, k \in \{j_\alpha\}} (x_j - x_k) \prod_{j, k \notin \{j_\alpha\}} (x_j - x_k). \tag{17}$$

Higher energy states can be obtained through the action of Q_+^α .

It remains to be shown that the *complete* spectrum is obtained under the action of the operators Q_0^α , Q_\pm^α on the ferromagnetic vacuum $|0\rangle$. It is conceivable that successive applications of different sets of these operators leads to the same final state. For a complete answer to this question the algebra of the operators Q_0^α , Q_\pm^α needs further study. However, a simple example shows that in general different states are generated by different operator sequences in the Q_0^α - E plane (figure 1): starting with the one-magnon state $|\psi^{(1)}\rangle$ (equation (14)) one obtains

$$|a; 2\rangle = Q_0^- Q_-^z |\psi^{(1)}\rangle = Q_0^- |\psi^{(2)}\rangle \propto \sum_{j \neq k} (x_j^2 + x_k^2 - (N-1)) J_j^- J_k^- |0\rangle. \tag{18}$$

A different state with the same energy and magnetization can be constructed from $|\psi^{(1)}\rangle$ through the action of Q_-^- resulting in

$$|b; 2\rangle = \left(\sum_j x_j J_j^- + 2 \sum_{j \neq k} w_{jk} J_j^z J_k^- \right) |\psi^{(1)}\rangle = \sum_{j \neq k} (2x_j x_k + 1) J_j^- J_k^- |0\rangle. \tag{19}$$

It is straightforward to show that $|a; 2\rangle$ and $|b; 2\rangle$ are indeed linearly independent. Note that they are *not* orthogonal though.

Finally, we observe that the simple structure of the spectrum found from the numerical investigation of finite chains allows for an alternative description through an effective single particle Hamiltonian with additional energy proportional to the square of the total number of particles

$$\mathcal{H}^{\text{eff}} = \sum_{k=1}^N (1-k)n_k - \sum_{k < k'} n_k n_{k'} + \left(\frac{1}{2}N\right)^2. \tag{20}$$

The occupation numbers n_k take values 0, 1. From (20) an expression for the partition function of the spin chain can be obtained

$$\mathcal{Z} = \sum_{k=0}^N z^{(N/2-k)^2} \prod_{r=1}^k \frac{1 - z^{N-r+1}}{1 - z^r} \quad z = e^{-\beta} \tag{21}$$

with the leading terms for low temperatures (and sufficiently large N)

$$\mathcal{Z} = 1 + 3z + 4z^2 + 7z^3 + 13z^4 + O(z^5). \tag{22}$$

To perform the thermodynamic limit ($N \rightarrow \infty$) in a meaningful way the energies have to be rescaled by a factor of N . This makes the excitations in the infinite system massless, hence the model has a critical point at $T = 0$. In this limit (21) can be brought into a closed form. The resulting expression for the free energy per spin reads

$$\frac{\mathcal{F}}{N} = -T^2 \left(\frac{\pi^2}{6} + 2f(1 + e^{-\beta/2}) \right) \quad (23)$$

where $f(x)$ is the dilogarithm

$$f(x) = - \int_x^1 dt \frac{\ln t}{1-t}. \quad (24)$$

The asymptotic behaviour of \mathcal{F} at low and high temperatures is given by

$$\frac{\mathcal{F}}{N} \sim \begin{cases} -T^2 \left(\frac{1}{6} \pi^2 - 2e^{-\beta/2} + o(e^{-\beta}) \right) & \text{for } T \rightarrow 0 \\ -T \ln 2 + \frac{1}{8} + o(T^{-1}) & \text{for } T \rightarrow \infty. \end{cases} \quad (25)$$

From (23) further thermodynamic quantities are easily extracted.

In this letter I have studied a new integrable model in the class of spin chains related to the classical Calogero–Sutherland models. As in other lattice models with inverse square exchange together with the integrability one finds a very simple and highly degenerate spectrum. In the model investigated here this can be understood in terms of the existence of an algebra of raising and lowering operators (11). It remains to be studied whether these operators can be interpreted in a similar way to the corresponding ones in the periodic Haldane–Shastry spin chain, which have been shown to be the level-one generators of a Yangian. A detailed understanding of this algebra is also necessary to prove the equivalence of the original Hamiltonian (1) and the effective one (20) and may lead to a rigorous foundation for the asymptotic Bethe ansatz in the integrable models with inverse square exchange.

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